Spectrum and Essential Spectrum of Toeplitz Operators

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Spectrum

The spectrum $\sigma(T)$ of a bounded linear operator T acting on a Hilbert space H is the set of complex numbers λ such that $\lambda I - T$ does not have an inverse that is a bounded linear operator.

If $H = C^n$, T can be viewed as a matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

and so its spectrum consists of eigenvalues of the matrix. But if H is an infinite dimensional Hilbert space, the spectrum of its bounded operator T may have more numbers than its eigenvalues $\sigma_p(T)$.

Essential Spectrum

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$$\sigma_e(T) = \{\lambda \in C : [\lambda I - T] \text{ is not invertible in } \mathcal{C}(H).\}$$

Calkin algebra $\mathcal{C}(H) = B(H)/K(H)$
$B(H)$: the algebra of bounded linear operators on H.
$K(H)$: the ideal of compact operators on H.

Fredholm Index and Spectral Picture

If λ is not in $\sigma_e(T)$, $T - \lambda I$ is Freholm. The Fredholm index is defined by

$$ind(T - \lambda I) = \dim Ker(T - \lambda I) - \dim Ker(T - \lambda I)^*.$$

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Carl Pearcy, Some Recent developments in Operator theory, CBMS 36, 1975.

Theorem

Let Ω be a connected component of $C \backslash \sigma_e(T)$ such that $ind(T - \lambda I) = 0$ for each $\lambda \in \Omega$. Then one of the following holds:
(a) $\Omega \cap \sigma(T)$ is empty.
(b) $\Omega \subset \sigma(T)$.
(c) $\Omega \cap \sigma(T)$ is a countable set of isolated eigenvalues of T, each having finite multiplicity.
Furthermore the intersection of $\sigma(T)$ with the unbounded component of $C \backslash \sigma_e(T)$ is a countable set of isolated eigenvalues of T, each of which has finite multiplicity.

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Toeplitz Operators on the Hardy space

A Toeplitz operator on the Hardy space is the compression of a multiplication operator on the circle to the Hardy space

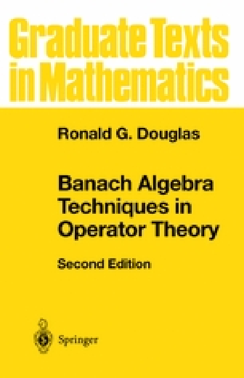
Let ∂D be the circle, with the standard Lebesgue measure, and $L^2(\partial D)$ be the Hilbert space of square-integrable functions. A bounded measurable function ϕ on ∂D defines a multiplication operator M_ϕ on $L^2(\partial D)$. Let P be the projection from $L^2(\partial D)$ onto the Hardy space H^2. The Toeplitz operator with symbol ϕ is defined by

$$T_\phi = PM_\phi|_{H^2}$$

Toeplitz matrix

A bounded operator on H^2 is Toeplitz if and only if its matrix representation, in the basis $\{z^n\}_0^\infty$, has constant diagonals:

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \end{bmatrix}$$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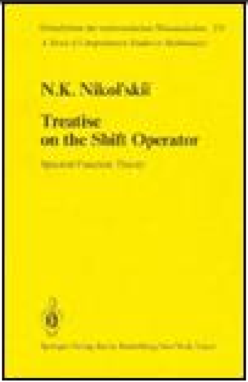
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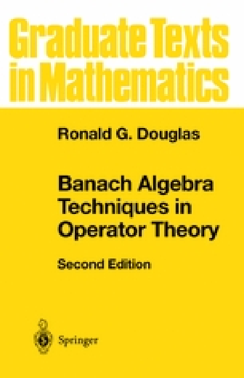
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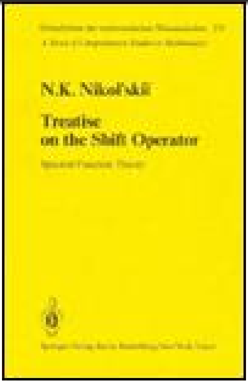
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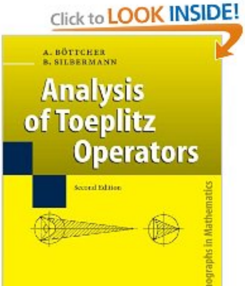
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Analysis of Toeplitz Operators
Second Edition

Bergman space

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The *Bergman space* L^2_a is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$:

Bergman space

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The *Bergman space* L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty.$$

Bergman space

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The *Bergman space* L^2_a is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty.$$

Let $e_n = \sqrt{n+1} z^n$. Then $\{e_n\}_0^\infty$ form an orthonormal basis of the Bergman space L^2_a.

Toeplitz Operators

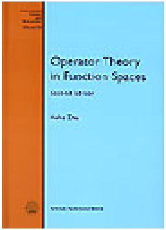
For $\phi \in L^\infty(\mathbb{D}, dA)$ where dA is normalized area measure on $\mathbb{D}$, the *Toeplitz operator* T_ϕ with symbol ϕ is the operator on L^2_a defined by

$$T_\phi f = P(\phi f);$$

here P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L^2_a. Note that if $\phi \in H^\infty$ (the set of bounded analytic functions on $\partial\mathbb{D}$), then T_ϕ is just the operator of multiplication by ϕ on L^2_a.

Operator Theory in Function Spaces

Kehe Zhu



Let $e_n = \sqrt{n+1}z^n$ and $\phi(z) = \sum_{j=-\infty}^{-1} a_j \bar{z}^{|j|} + \sum_{j=0}^{\infty} a_j z^j$.

$$\langle T_\phi e_i, e_j\rangle = \sqrt{i+1}\sqrt{j+1}a_{j-i}\langle z^j, z^j\rangle = a_{j-i}\sqrt{\frac{i+1}{j+1}}.$$

On the basis $\{e_n\}$, the matrix representation of the Toeplitz operator T_ϕ on the Bergman space is given by

$$\begin{bmatrix} a_0 & \sqrt{\frac{2}{1}}a_{-1} & \sqrt{\frac{3}{1}}a_{-2} & \sqrt{\frac{4}{1}}a_{-3} & \cdots & \cdots \\ \sqrt{\frac{1}{2}}a_1 & a_0 & \sqrt{\frac{3}{2}}a_{-1} & \sqrt{\frac{4}{2}}a_{-2} & \cdots & \cdots \\ \sqrt{\frac{1}{3}}a_2 & \sqrt{\frac{2}{3}}a_1 & a_0 & \sqrt{\frac{4}{3}}a_{-1} & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \end{bmatrix}.$$

Some algebraic properties

(a) $T_{\alpha\phi+\beta\psi} = \alpha T_\phi + \beta T_\psi$.

(b) If ϕ is in H^∞, then

$$T_\psi T_\phi = T_{\psi\phi}.$$

(c) If $\overline{\psi}$ is in H^∞, then

$$T_\psi T_\phi = T_{\psi\phi}.$$

(d) $T_\phi^* = T_{\overline{\phi}}$.

(e) If $\phi \geq 0$, then $T_\phi \geq 0$.

Fredholm index for Toeplitz operator

If ϕ is continuous on the unit circle ∂D and does not vanish on ∂D, then T_ϕ is Fredholm and

$$ind(T_\phi) = n(\phi(\partial\mathbb{D}), 0).$$

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If ϕ is continuous on the unit circle ∂D and does not vanish on ∂D, then T_ϕ is Fredholm and

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For a closed curve γ in the complex plane $\mathbb{C}$ and $a \in \mathbb{C} \backslash \gamma$, define the winding number of the curve γ with respect to a to be

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - a}.$$

Bergman shift

On the basis $\{e_n = \sqrt{n+1}z^n\}$, the Toeplitz operator T_z with symbol z is a weighted shift operator, called the Bergman shift:

$$T_z e_n = \sqrt{\frac{n+1}{n+2}}e_{n+1},$$

and hence $T_{\bar{z}}$ is a backward weighted shift:

$$T_{\bar{z}} e_n = \begin{cases} 0 & n = 0 \\ \sqrt{\frac{n}{n+1}}e_{n-1}. & n > 0 \end{cases} \tag{1}$$

The matrix representation of the Toeplitz operators $T_{1-|z|^2} = I - T_z^* T_z$ is given by

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdots & \cdots \\ 0 & \frac{1}{3} & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \frac{1}{4} & 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \end{bmatrix}$$

Differences between H^2 and L^2_a

Theorem (Coburn Theorem)

If $T_\phi \neq 0$ on the Hardy space, either $\ker T_\phi = \{0\}$ or $\ker T_\phi^* = \{0\}$.

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Does Coburn theorem hold on the Bergman space?

No! On the Bergman space, both $\ker T_{1-|z|^2-\frac{1}{2}}$ and $\ker T^*_{1-|z|^2-\frac{1}{2}}$ contain the function 1.

But $1-|z|^2-\frac{1}{2}$ is not harmonic in the unit disk and $T_{1-|z|^2}$ is compact!

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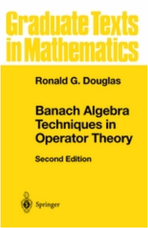
Theorem (Douglas Theorem)

The essential spectrum $\sigma_e(T_\phi)$ of a Toeplitz operator on the Hardy space is also connected.



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Compact Toeplitz operators on the Hardy space and the Bergman space

Theorem

On the Hardy space, T_ϕ is compact if and only if $\phi = 0$.

Theorem (Axler-Zheng)

For $\phi \in L^\infty(D)$, T_ϕ is compact on the Hardy space if and only if

$$\lim_{|z|\to 1} \int_D \phi(w) \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} dA(w) = 0.$$

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There is no nontrivial compact Toeplitz operator with bounded harmonic symbol on the Bergman space.

Revised Questions

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Is the spectrum $\sigma(T_\phi)$ of a Toeplitz operator on the Bergman space connected if ϕ is bounded and harmonic on the unit disk?

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Sundberg's conjecture: Yes! (Problem 7.10 in V.P. Havin and N.K. Nikolski (Eds), Linear and Complex Analysis Problem Book 3, Lecture notes in Mathematics 1573, 1994).

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TOEPLITZ OPERATORS ON THE BERGMAN SPACE

C. Sundberg

Let A^2 denote the Bergman space of analytic functions in $L^2(\mathbb{D})$, and let P be the orthogonal projection of $L^2(\mathbb{D})$ onto A^2. For $\varphi \in L^\infty(\mathbb{D})$ we define the Toeplitz operator with symbol φ by $T_\varphi = P(\varphi f)$. In general the behaviour of these operators may be quite different from that of the Toeplitz operators on the Hardy space H^2. However it is shown in [1] that Toeplitz operators on A^2 with *harmonic* symbols behave quite similarly to those on H^2, and one can prove analogues for this class of many results about Toeplitz operators on H^2.

An important result about Toeplitz operators on H^2 is Widom's Theorem, which states that the spectrum of such an operator is connected ([2]). This suggests our problem.

Conjecture. *A Toeplitz operator on A^2 with harmonic symbol has a connected spectrum.*

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This was an open question in (G. McDonald and C. Sundberg, *Indiana Univ. Math. J.* 28 (1979)).

Let ϕ be in $H^{\infty}(D)$.

Supports for Sundberg's conjecture

Let ϕ be in $H^\infty(D)$.

If λ is not in the closure of $\phi(D)$, then $\frac{1}{\phi - \lambda}$ is in $H^\infty(D)$ and

$$T_{\phi-\lambda} T_{(\phi-\lambda)^{-1}} = T_{(\phi-\lambda)^{-1}} T_{\phi-\lambda} = I.$$

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If $\lambda = \phi(a)$ for some $a \in D$, then

$$T^*_{\phi-\lambda}k_a = 0.$$

Hence
(a) If ϕ is analytic on the unit disk, then

$$\sigma(T_\phi) = clos\phi(D).$$

(b) If ϕ is real and harmonic on the unit disk, then

$$\sigma(T_\phi) = [\inf \phi, \sup \phi].$$

Supports for Sundberg's conjecture

(b) If ϕ is real and harmonic on the unit disk, then

$$\sigma(T_\phi) = [\inf \phi, \sup \phi].$$

(c) If ϕ is harmonic and has piecewise continuous boundary values, then $\sigma_e(T_\phi)$ consists of the path formed boundary values of ϕ by joining the one-sided limits at discontinuities by straight line segments and hence $\sigma_e(T_\phi)$ is connected.

(b) If ϕ is real and harmonic on the unit disk, then

$$\sigma(T_\phi) = [\inf \phi, \sup \phi].$$

(c) If ϕ is harmonic and has piecewise continuous boundary values, then $\sigma_e(T_\phi)$ consists of the path formed boundary values of ϕ by joining the one-sided limits at discontinuities by straight line segments and hence $\sigma_e(T_\phi)$ is connected.

(b) and (c) are contained in (G. McDonald and C. Sundberg, *Indiana Univ. Math. J.* 28 (1979)).

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(d) 0 **is an isolated eigenvalue of** T_h**.**

Eigenvectors of T_h for the eigenvalue 0

Let f be an eigenvector for T_h for the eigenvalue 0. Then

$$\begin{aligned} 0 &= T_h f(z) \\ &= T_{\bar{z}} f(z) + T_{\phi(z)} f(z) \\ &= \frac{1}{z^2} \int_0^z w f'(w) dw + \phi(z) f(z). \end{aligned}$$

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Lemma

For f in the Bergman space L_a^2,

$$T_{\bar{z}} f(z) = \frac{1}{z^2} \int_0^z w f'(w) dw. \tag{2}$$

$\frac{1}{z^2}\int_0^z wf'(w)dw + \phi(z)f(z) = 0$

This is equivalent to the following first order differential equation

$$(1 + z\phi(z))f'(z) = -(2\phi(z) + z\phi'(z))f(z). \qquad (3)$$

For a fixed $0 < r < 1$, we want
(a) a rational function $\eta(z)$ with poles outside the closure of the unit disk such that

$$2\phi(z) + z\phi'(z) = (z - r)\eta(z);$$

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(b) $1 + z\phi(z)$ has a simple zero at $z = r$ and no other zeros in $\overline{\mathbb{D}}$.
Write

$$\psi(z) = \frac{1 + z\phi(z)}{z - r}.$$

Then ψ is a rational function with poles outside of the closure of the unit disk.

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(3) becomes

$$\frac{f'(z)}{f(z)} = -\frac{\eta(z)}{\psi(z)}.$$

$(1 + z\phi(z))f'(z) = -(2\phi(z) + z\phi'(z))f(z)$

A solution of the above equation in the Bergman space L_a^2 is given by

$$f(z) = exp[-\int_0^z \frac{\eta(w)}{\psi(w)} dw].$$

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$$dim \quad ker T_h = dim \quad ker T_h^*.$$

Lemma

For each $0 < r < 1$*, there exists a rational function* $\phi(z)$ *with poles outside* $\overline{\mathbb{D}}$ *such that*

(a) $2\phi(r) + r\phi'(r) = 0.$

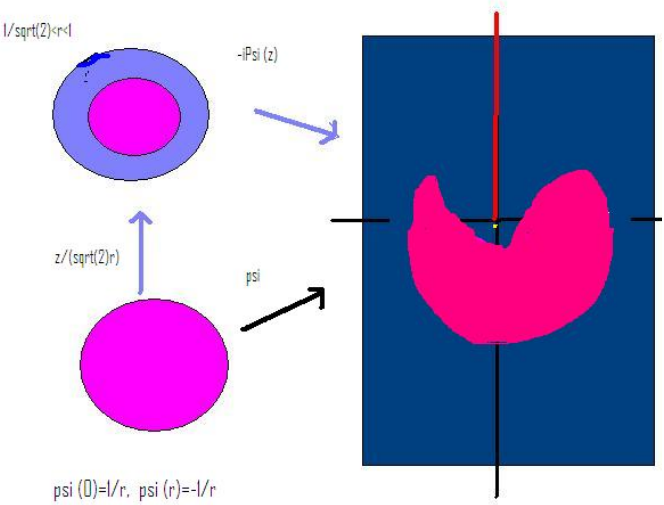
(b) $1 + z\phi(z)$ *has a simple zero at* $z = r$ *and no other zeros in* $\overline{\mathbb{D}}$*.*

(c) *The winding number*

$$n(h(\partial\mathbb{D}), 0) = 0$$

where $h = \overline{z} + \phi(z)$*.*

Sketch of Proof



Disconnected Spectrum

Theorem

Let $h(z) = \bar{z} + \phi(z)$ *Then* 0 *is an eigenvalue of* T_h *and is an isolated point of* $\sigma(T_h)$*. Hence* $\sigma(T_h)$ *is disconnected.*

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(2) $0 \in \sigma_p(T_h) \cap \Omega$ where

$$\begin{aligned} \Omega &= \{\lambda \notin \sigma_e(T_h) : ind(T_h - \lambda I) = 0\} \\ &= \{\lambda \notin h(\partial\mathbb{D}) : n(h(\partial\mathbb{D}), \lambda) = 0\}. \end{aligned}$$

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Sketch of the proof
Want: $\Omega \cap \sigma_p(T_h)$ is countable.

Cauchy's argument principle

if $f(z)$ is a meromorphic function inside and on some closed contour C, and f has no zeros or poles on C, then

$$\oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi i(N - P)$$

$\lambda \in \Omega \cap \sigma_p(T_h),\ n(h(\partial\mathbb{D}), \lambda) = 0$

Since $\frac{1}{z} + \phi(z)$ has a simple pole at $z = 0$ and no other poles in the unit disk $\mathbb{D}$, the argument principle tells us that if λ is in $\Omega \cap \sigma_p(T_h)$, there is a unique point z_λ in $\mathbb{D}$ such that

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As λ is an eigenvalue of T_h, there is a nonzero function g in the Bergman space L_a^2 such that

$$\begin{aligned} \lambda g &= T_h g(z) \\ &= T_{\bar{z}} g(z) + T_{\phi(z)} g(z) \\ &= \frac{1}{z^2} \int_0^z w g'(w) dw + \phi(z) g(z). \end{aligned}$$

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We solve the above equation to obtain

$$\frac{g'(z)}{g(z)} = -\frac{2(\phi(z) - \lambda) + z\phi'(z)}{1 + z(\phi(z) - \lambda)}.$$

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$$-\frac{2(\phi(z_\lambda) - \lambda) + z_\lambda\phi'(z_\lambda)}{\phi(z_\lambda) - \lambda + z_\lambda\phi'(z_\lambda)} = -1 - \frac{1}{1 - z_\lambda^2\phi'(z_\lambda)}.$$

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The regularity of $g(z)$ at $z = z_\lambda$ forces this residue to be in $\mathbb{N} = \{0, 1, 2, 3, \cdots,\}$ which leads to

$$z_\lambda^2\phi'(z_\lambda) = 1 + \frac{1}{n+1}$$

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to be a countable set. Thus 0 is an isolated point in $\sigma(T)$. Hence we conclude that the spectrum $\sigma(T_h)$ is disconnected.

Unitary operator U_z

For $z \in \mathbb{D}$, let ϕ_z be the analytic map of $\mathbb{D}$ onto $\mathbb{D}$ defined by

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}. \tag{4}$$

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For S a bounded operator on L^2_a, define S_z to be the bounded operator on L^2_a given by conjugation with U_z:

$$S_z = U_z S U_z.$$

Let $\mathcal{M}$ be the maximal ideal space of H^∞, i.e., the set of complex homomorphisms of H^∞ with w^*-topology. Then $\mathcal{M}$ is a compact Hausdorff space.

Maximal ideal space of H^∞

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$$\lim_{z \to m} \alpha_z = \beta$$

means (as you should expect) that for each open set X in E containing β, there is an open set Y in $\mathcal{M}$ containing m such that $\alpha_z \in X$ for all $z \in Y \cap \mathbb{D}$. Note that with this notation z is always assumed to lie in $\mathbb{D}$.

Hoffman map

For $m \in \mathcal{M}$, let $\phi_m \colon \mathbb{D} \to \mathcal{M}$ denote the *Hoffman map*. This is defined by setting

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for $w \in \mathbb{D}$; here we are taking a limit in $\mathcal{M}$. The existence of this limit, as well as many other deep properties of ϕ_m, was proved by Hoffman (*Ann. Math.*, **103** (1967)).

Localization S_m of S in Toeplitz algebra

The *Toeplitz algebra* $\mathcal{T}$ is the C^*-subalgebra of $\mathcal{B}(L^2_a)$ generated by $\{T_g : g \in H^\infty\}$.

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Lemma

If $S \in \mathcal{T}$, the Toeplitz algebra and $m \in \mathcal{M}$, then there exists $S_m \in \mathcal{T}$ such that

$$\lim_{z \to m} \|S_z f - S_m f\| = 0$$

for every f in L^2_a. If $S = T_{u_1} \dots T_{u_n}$, where $u_1, \dots, u_n \in \mathcal{U}$, then $S_m = T_{u_1 \circ \phi_m} \dots T_{u_n \circ \phi_m}$.

Using a similar argument as one in the proof of Theorem 10.3 in (D. Suarez, *Indiana Univ. Math. J.*, **56** (2007)), we have the following theorem.

THEOREM

If $S \in \mathcal{T}$, *the Toeplitz algebra, then*

$$\mathbb{C}\backslash\sigma_e(S) = \{\lambda \in \mathbb{C} : \lambda \notin \bigcup_{m\in\mathcal{M}\backslash\mathbb{D}} \sigma(S_m) \quad \textit{and}$$

$$\sup_{m\in\mathcal{M}\backslash\mathbb{D}} \|(S_m - \lambda I)^{-1}\| < \infty\}.$$

Thin Blaschke product

To a sequence $\{z_n\}_n$ in $\mathbb{D}$ with $\sum_{n=1}^{\infty}(1-|z_n|)<\infty$, there corresponds a Blaschke product

$$b(z)=\prod_{n=1}^{\infty}\frac{-\overline{z}_n}{|z_n|}\frac{z-z_n}{1-\overline{z}_n z}, \quad z\in\mathbb{D}.$$

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A sequence $\{z_n\}_n$ and its associated Blaschke product are called thin if

$$\lim_{n\to\infty}\prod_{k\neq n}\left|\frac{z_n-z_k}{1-\bar{z}_k z_n}\right|=1.$$

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$$\lim_{n\to\infty} \prod_{k\neq n} \left| \frac{z_n - z_k}{1 - \overline{z}_k z_n} \right| = 1.$$

Hedenmalm (*Proc. Amer. Math. Soc.*, 99 (1987)) showed that for each m in $\mathcal{M}\backslash\mathbb{D}$, either

$$b \circ \phi_m(z) = \lambda_m \quad or \quad b \circ \phi_m(z) \in Aut(\mathbb{D})$$

for some unimodular constant λ_m. The latter case actually occurs if m is in the Gleason part of some point in the closure of zeros of b in $\mathbb{D}$.

Theorem

Let F be a continuous function on the closure $\overline{\mathbb{D}}$ of the unit disk, b be an infinite thin Blaschke product and $F_b = F \circ b$. Then

$$\sigma_e(T_{F_b}) = \sigma(T_F).$$

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Proof Let $S = T_{F_b}$.
For each m in $\mathcal{M} \backslash \mathbb{D}$,

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Proof Let $S = T_{F_b}$.
For each m in $\mathcal{M} \backslash \mathbb{D}$,

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By Hedenmalm's result above, we have that for each m in $\mathcal{M} \backslash \mathbb{D}$, either
(a) $b \circ \phi_m(z) = \lambda_m$ for some unimodular constant λ_m or
(b) $\tau_m = b \circ \phi_m(z) \in Aut(\mathbb{D})$.

(A) $b \circ \phi_m(z) = \lambda_m$

S_m equals the operator $F(\lambda_m)I$ and hence $\sigma(S_m)$ equals one point $F(\lambda_m)$. Thus

$$\sigma(S_m) \subset F(\partial\mathbb{D}) \subset \sigma(T_F),$$

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$$\sigma(S_m) \subset F(\partial\mathbb{D}) \subset \sigma(T_F),$$

and for each λ not in $\sigma(T_F)$,

$$\begin{aligned} \|(S_m - \lambda I)^{-1}\| &= \frac{1}{|F(\lambda_m) - \lambda|} \\ &\leq \frac{1}{dis(\lambda, \sigma(T_F))}. \end{aligned}$$

(B) $\tau_m = b \circ \phi_m(z) \in Aut(\mathbb{D})$

$$\begin{aligned} S_m &= T_{F \circ \tau_m} \\ &= V_m T_F V_m^* \end{aligned}$$

where V_m is the unitary operator on the Bergman space L_a^2 given by

$$V_m f(z) = f(\tau_m(z)) \tau_m'(z).$$

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Thus $\sigma(S_m) = \sigma(T_F)$ and for each λ in $\mathbb{C}\backslash\sigma(S_m)$,

$$\begin{aligned} \|(S_m - \lambda I)^{-1}\| &= \|V_m T_{F-\lambda}^{-1} V_m^*\| \\ &= \|T_{F-\lambda}^{-1}\|. \end{aligned}$$

So we have

$$\cup_{m \in \mathcal{M} \setminus \mathbb{D}} \sigma(S_m) = \sigma(T_F)$$

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By the following theorem

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If $S \in \mathcal{T}$, the Toeplitz algebra, then

$$\mathbb{C} \setminus \sigma_e(S) = \{\lambda \in \mathbb{C} : \lambda \notin \bigcup_{m \in \mathcal{M} \setminus \mathbb{D}} \sigma(S_m) \quad \textit{and}$$

$$\sup_{m \in \mathcal{M} \setminus \mathbb{D}} \|(S_m - \lambda I)^{-1}\| < \infty\}.$$

we have that

$$\sigma_e(T_{F_b}) = \sigma_e(S) = \sigma(T_F).$$

Disconnected Essential Spectrum

Theorem

Let h be $\bar{z} + \phi$ such that $\sigma(T_h)$ is disconnected. Let b be an infinite thin Blaschke product and $h_b = h \circ b$. Then

$$\sigma_e(T_{h_b}) = \sigma(T_h)$$

is disconnected.

Lemma

For each $0 < r < 1$*, there exists a rational function* $\phi(z)$ *with poles outside* $\overline{\mathbb{D}}$ *such that*

(a) $2\phi(r) + r\phi'(r) = 0.$

(b) $1 + z\phi(z)$ *has a simple zero at* $z = r$ *and no other zeros in* $\overline{\mathbb{D}}$.

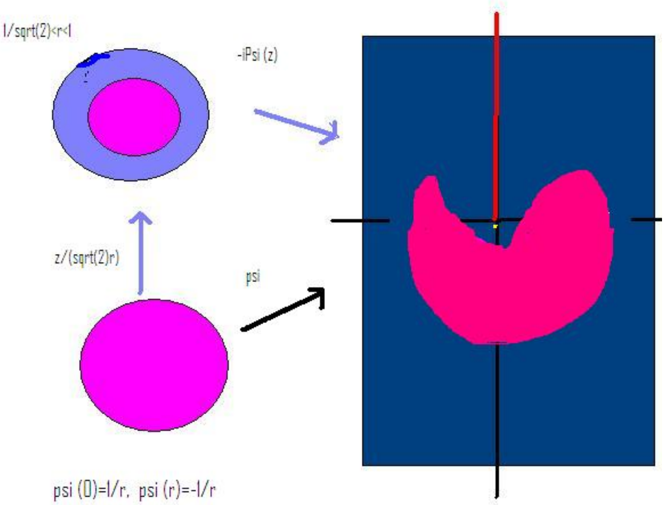
(c) *The winding number*

$$n(h(\partial\mathbb{D}), 0) = 0$$

where $h = \overline{z} + \phi(z)$.

Proof: For $\frac{1}{\sqrt{2}} < r < 1$, we are going to construct ϕ by some conformal mappings.

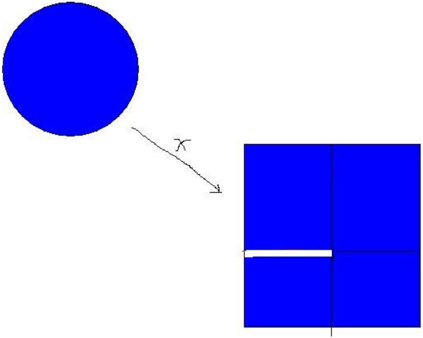
Proof of Lemma



Proof of Lemma

Let λ be the unimodular constant $i\frac{2+i}{2-i}\frac{\sqrt{2}}{1+i}$. Define

$$\chi(z) = \frac{1}{2r}\left(\frac{1+z}{1-z}\right)^2,$$

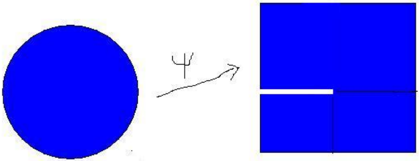


Let

$$\Psi(z) = \chi\left(\frac{\lambda z - \frac{i}{2-i}}{1 + \frac{i}{2+i}\lambda z}\right).$$

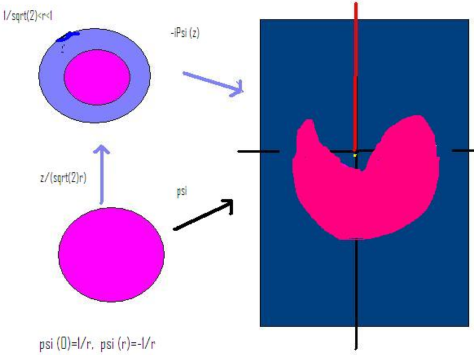
Then

$$\Psi(0) = -\frac{i}{r}, \quad \Psi\left(\frac{1}{\sqrt{2}}\right) = \frac{i}{r}.$$



Now define

$$\psi(z) = -i\Psi(\frac{1}{\sqrt{2}r}z).$$



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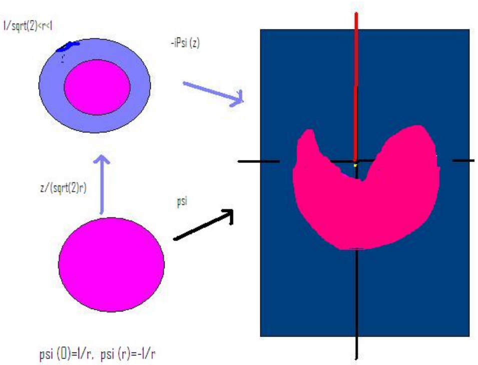

Since $r > \frac{1}{\sqrt{2}}$, the poles of $\psi(z)$ are outside $\overline{\mathbb{D}}$.

Since χ is a conformal map of $\mathbb{D}$ onto $\mathbb{C} \backslash (-\infty, 0]$, ψ is a conformal map of $\mathbb{D}$ onto a region bounded by a simple closed curve and 0 is outside the region. In particular $\psi(\partial \mathbb{D})$ does not wind around 0 and $\psi(z) \neq 0$ for all z in $\overline{\mathbb{D}}$.

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Defining

$$\phi(z) = \frac{(z-r)\psi(z) - 1}{z},$$

we see that (a) and (b) are satisfied:

(a) $2\phi(r) + r\phi'(r) = 0$.

(b) $1 + z\phi(z)$ has a simple zero at $z = r$ and no other zeros in $\overline{\mathbb{D}}$.

On $\partial\mathbb{D}$

$$\begin{aligned}\bar{z} + \phi(z) &= \frac{1}{z} + \phi(z) \\ &= \frac{1 + z\phi(z)}{z} \\ &= \frac{z - r}{z}\psi(z).\end{aligned}$$

So (c) is satisfied too.

PROOF OF $T_{\bar{z}}f(z) = \frac{1}{z^2}\int_0^z wf'(w)dw$

LEMMA

For f in the Bergman space L_a^2,

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Proof. Note that

$$\{e_n = \sqrt{n+1}z^n\}_{n=0}^{\infty}$$

is an orthonormal basis of the Bergman space. To prove this lemma, we need only verify the above equality for each $f(z) = e_n$.

Proof of $T_{\bar{z}} f(z) = \frac{1}{z^2} \int_0^z w f'(w) dw$

Lemma

For f in the Bergman space L_a^2,

$$T_{\bar{z}} f(z) = \frac{1}{z^2} \int_0^z w f'(w) dw.$$

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is an orthonormal basis of the Bergman space. To prove this lemma, we need only verify the above equality for each $f(z) = e_n$. As $T_{\bar{z}}$ is the adjoint of the Bergman shift, we have

$$T_{\bar{z}} e_n = \begin{cases} 0 & n = 0 \\ \sqrt{\frac{n}{n+1}} e_{n-1}. & n > 0 \end{cases}$$

On the other hand, since $e_n(w) = \sqrt{n+1}w^n$, an easy calculation gives

$$\int_0^z w e_n'(w) dw \quad = \quad \frac{n z^{n+1}}{\sqrt{n+1}}$$

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Thus we have

$$\begin{aligned} \frac{1}{z^2}\int_0^z w e_n'(w) dw \quad &= \quad \frac{nz^{n-1}}{\sqrt{n+1}} \\ &= \quad \sqrt{\frac{n}{n+1}} e_{n-1}, \end{aligned}$$

to obtain

$$T_{\bar{z}} e_n = \frac{1}{z^2}\int_0^z w e_n'(w) dw.$$

This completes the proof of the lemma.